Lecture 8: Cross Sections and Decay Rates

Sept 20, 2016

Formalism follows the discussion in Halzen and Martin

Overview

- Problems in particle physics fall into 3 categories
    1. Descriptions of bound states
    2. Decays
    3. Scattering
- All have QM analogs
- But need to extend to include relativity (Lorentz invariant form)
- Today:
    - Review definitions and terminology
    - Show some simple examples

Relativity Review

- x^μ is a 4-vector (0=time, 1-3=space)
- Lorentz boost long z-axis:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu$$

repeated indices imply sum.

- Metric $g_{\mu\nu}$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Units: $\hbar = c = 1$
- $g_{\mu\nu}x^\mu x^\nu$ is Lorentz invariant for any x^μ, x^ν
- Derivatives: $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$
  - $\partial_\mu = (\partial/\partial t, \vec{\nabla})$,
  - $\partial^\mu = (\partial/\partial t, -\vec{\nabla})$

  Thus $p^\mu \Rightarrow i\partial^\mu$
- D'Alembertian $\Box^2 \equiv \partial_\mu \partial^\mu$
- Dot product

$$a \cdot b = a_\mu b^\mu = a_0 b_0 - \vec{a} \cdot \vec{b}$$

The Klein-Gordon Equation (QED without spin)

- Relativistic energy-momentum conservation

$$E^2 = p^2 + m^2$$

becomes in operator form for a wave function ϕ

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$

- Multiply by ϕ^* on the right and then subtract complex conjugate equation

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\left[i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)\right] + \nabla \cdot \left[-i\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right)\right] \\ &= \frac{\partial \rho}{\partial t} + \nabla \cdot j \end{aligned}$$

The continuity equation (for probability)

- Using free particle solutions $\phi = N e^{ip \cdot x - iEt}$ we identify

$$\begin{aligned} \rho &\equiv 2E|N|^2 \\ \vec{j} &\equiv 2p|N|^2 \\ j^\mu &\equiv (\rho, \vec{j}) \end{aligned}$$

Klein-Gordon equation: Negative Energy Solutions

- Since $E = \pm\sqrt{p^2 + m^2}$, negative energy solutions exist
  - $\rho \equiv 2E|N|^2 \rightarrow$ negative probability density ρ
- Resolve this problem by multiplying by charge q
- Now j^μ is a four-vector current
  - The negative energy states are redefined as states of opposite electric charge

Introduction of relativity requires introduction of anti-particle states!

Some Comments on Wave Function Normalization

- From page 3, for plane wave states:

$$\begin{aligned} \rho &= 2E|N|^2 \\ \int \rho dV &= 2E|N|^2 V \end{aligned}$$

- Now boost long x-axis:

$$\begin{aligned} d^3x &\rightarrow \frac{1}{\gamma} d^3 \\ E &\rightarrow \gamma E \end{aligned}$$

- If we define wave function normalization $|N|^2 \equiv 1/V$ then # of particles (or total charge) is $2E$ independent of frame:

$$\int \rho dV = 2E$$

This is the standard normalization

Dirac Equation: Adding Spin

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

where

$$\begin{aligned}
\gamma_\mu &= (\beta, \beta\vec{\alpha}) \\
\beta &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
\vec{\alpha} &= \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}
\end{aligned}$$

- 4-component spinors (<u>not</u> a 4-vector!)
- Four free particle solutions
  - Two particle states (helicity)
  - Reinterpret negative energy states as two antiparticle states
- $j^\mu = -e\overline{\psi}\gamma^\mu\psi$: conserved charge and current

Perturbation Theory (I) Non-relativistic reminder

- Potential $\mathsf{V}(x,t)$ limited to finite spatial extent
- Assume $\mathsf{V}(x,t)$ small so PT works
- ϕ_n are solns to $H_0\phi_n = E_n\phi_n$
- For $H = H_0 + \mathsf{V}(x,t)$:

$$\begin{aligned}(H_0 + \mathsf{V}(x,t))\,\psi &= -i\frac{\partial\psi}{\partial t}, \\ \psi &= \sum_n a_n(t)\phi_n(x)e^{-iE_nt}\end{aligned}$$

therefore

$$i\sum_n \frac{da_n(t)}{dt}\phi_n(x)e^{-iE_nt} = \sum_n \mathsf{V}(x,t)a_n\phi_n(x)e^{-iE_nt}$$

- Multiply by ϕ^* and integrate:

$$i\frac{da_f}{dt}e^{-iE_nt} = -i\sum_n\int \mathsf{V}(x,t)a_n(t)\phi_f^*\phi_n e^{-iE_nt}d^3x$$

$$\frac{da_f}{dt} = -i\sum_n\int a_n(t)\mathsf{V}(x,t)\phi_f^*\phi_n e^{-i(E_n-E_f)t}d^3x$$

Perturbation Theory (II)

- Integrate over time from $-\mathsf{T}/2$ to $\mathsf{T}/2$
- At time $t = -\mathsf{T}/2$ in state i:

$$
\begin{aligned}
a_i(-\mathsf{T}/2) &= 1, \\
a_f(-\mathsf{T}/2) &= 0, \quad \text{for} \;\; n \neq i
\end{aligned}
$$

We find:

$$
\begin{aligned}
\frac{da_f}{dt} &= -i \int a_i(t) \mathsf{V}(x,t) \phi_f^* \phi_n e^{i(E_f - E_n)t} d^3x \\
&= -i \int \phi_f^* \mathsf{V}(x,t) \phi_n e^{i(E_f - E_n)t} d^3x \\
T_{fi} \equiv a_f(\mathsf{T}/2) &= -i \int_{-\mathsf{T}/2}^{T/2} \int \phi_f^* \mathsf{V}(x,t) \phi_n e^{i(E_f - E_n)t} d^3x dt'
\end{aligned}
$$

- Or in covarient form

$$
a_f = -i \int \phi_f^*(x) \mathsf{V}(x) \phi_i(x) d^4x
$$

Same expression holds for relativisitic QM

Perturbation Theory (III)

- If V has no time dependence

$$
\begin{aligned}
T_{fi} &= -i\mathsf{V}_{fi}\int_{-\infty}^{\infty} e^{i(E_f - Ei)t}dt \\
&= -2\pi i\mathsf{V}_{fi}\delta(E_f - E_i)
\end{aligned}
$$

  Conservation of energy

- Transition rate

$$
\begin{aligned}
w_{fi} &= \lim_{T\to\infty}\frac{|T_{fi}|^2}{\mathsf{T}} \\
&= \lim_{T\to\infty} 2\pi\frac{|\mathsf{V}_{fi}|^2}{\mathsf{T}}\delta(E_f - E_i)\int_{-\mathsf{T}}^{\mathsf{T}} e^{i(E_f - E_i)t}dt \\
&= \lim_{T\to\infty} 2\pi\frac{|\mathsf{V}_{fi}|^2}{\mathsf{T}}\delta(E_f - E_i)\int_{-\mathsf{T}}^{\mathsf{T}} dt \\
&= 2\pi|\mathsf{V}_{fi}|^2\delta(E_f - E_i)
\end{aligned}
$$

- Must integrate over all possible final states for a given initial state
  - Introduce density of states $\mathcal{D}(E_f)$

Fermi Golden Rule

- The non-relativistic result holds for relativistic case as well

$$w_{fi} = 2\pi |\mathsf{V}_{fi}|^2 \mathcal{D}(E_i)$$

where w_{fi} is the transition rate, V_{fi} is the "matrix element" and $\mathcal{D}(E_i)$ is the density of states factor, also called the phase space factor

- To lowest order

$$\mathsf{V}_{fi} = \int d^3\, x \phi_f^*(x) \mathsf{V}(x) \phi_i(x)$$

- To next order

$$\mathsf{V}_{fi} \to \mathsf{V}_{fi} + \sum_{b \neq i} \mathsf{V}_{fn} \frac{1}{E_i - E_n} \mathsf{V}_{ni}$$

and so forth for higher orders

- Relativistic phase space factor (before including any spin factors):

$$\text{No of final states/particle} = \frac{V d^3 p}{(2\pi)^3 2E}$$

The volume V always cancels out when we properly normalize the single particle wave functions $N = 1/\sqrt{V}$

More comments on next order in Perturbation Theory?

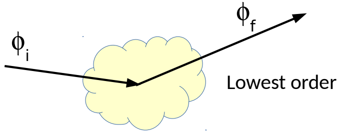
$$T_{fi} = T_{fi}^{lowest} - \sum_{n \neq i} \mathsf{V}_{fn} \mathsf{V}_{ni} \int_{-\infty}^{\infty} dt e^{i(E_f - E_n)t} \int_{-\infty}^{t} e^{i(E_n - E_i)t}$$

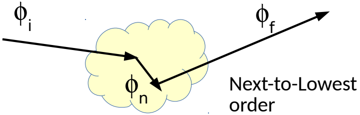
using $$\int dt' e^{i(E_n - E_i - i\epsilon)} = \frac{i e^{i(E_n - E_i - i\epsilon)t}}{E_i - E_n - i\epsilon}$$

$$T_{fi} = T_{fi}^{lowest} - 2\pi i \sum_{n \neq i} \frac{\mathsf{V}_{fn} \mathsf{V}_{ni}}{E_i - E_n - i\epsilon} \delta(E_f - E_i)$$

- Term in denominator is called the "'propagator factor"
- Intermediate states are virtual and don't have to conserve energy and momentum
- Overall δ-fn imposed energy conservation on result

The physical interpretation





Back to Fermi's Golden Rule

$$w_{fi} = 2\pi |\mathsf{V}_{fi}|^2 \mathcal{D}(E)$$

- Can divide transition rate into 2 partes:
  - Dynamics: in $|\mathsf{V}_{fi}|$
  - Kinematics: in $\mathcal{D}(E)$
- Have normalized wave function so $\int \rho dV = 2E$
  - But this is not Lorentz invariant
- Compensate by putting appropriate factor $(1/2E)$ into $\mathcal{D}(E)$
- From Stat Mech:

$$\mathcal{D}(E) = \frac{1}{h^3} \int d^3x d^3p = \frac{V}{(2\pi\hbar)^3} \int d^3p$$

- To make this Lorentz invariant

$$N \to \frac{V}{(2\pi\hbar)^3 2E} \int d^3p$$

and now set $\hbar = 1$

Calculating Cross Sections

- In all calculations arbitrary volume V cancels

$$\mathcal{D}(E) = \prod_{\text{final state particles}} \frac{d^3p}{(2\pi)^3 2E_f}$$

- $|T_{if}|^2$ includes δ-function to conserve energy-momentum
- Will concentrate on two special cases:
  1. Decay $A \to B + C$
  2. Scattering $A + B \to C + D$

Decays

- Decays

$$d\Gamma = \frac{1}{2E_A}|T_{if}|^2 \frac{d^3p_1}{(2\pi)^3 2E_1} \dots \frac{d^3p_n}{(2\pi)^3 2E_n}(2\pi)^4\delta^4(pA - p_1 - p_2 \dots p_n)$$

- For $A \rightarrow B + C$ in rest frame:

$$d\Gamma = \frac{p_f}{32\pi^2 m_A^2}\int |T_{fi}|^2 d\Omega$$

where p_f is momentum of one of the final state particles

Estimating Two-Body Decay Rates:

- Hadronic: eg ρ decay

  Dimensionally, T_{fi} is a mass or energy. Assume 1 GeV. Now estimate the width of $\rho(770) \to \pi\pi$. $p_{cm} \approx 360$ MeV.

  $$\Gamma = \frac{4\pi}{32\pi^2}\frac{360 \cdot 1000^2}{770^2}\mathrm{MeV} = 24\ \mathrm{MeV}$$

  True: 146 MeV. This is very good agreement!

- $\pi \to \mu\nu$

  This is a weak decay, so it has the Fermi constant in the amplitude: $G_F \approx 10^{-5}\mathrm{GeV}^{-2}$. So dimensionally,

  $$\Gamma \propto G_F^2 p^5$$

  If we guess that $p \approx m_\pi$ we get

  $$\Gamma = (10^{-5})^2(0.1)^5\ \mathrm{GeV} = 10^{-15}\mathrm{GeV}$$

π Decay Continued: Conversion Factors

Two convenient conversions, with $\hbar = c = 1$ are

$$1 = 197 \text{ MeV fermi}; \qquad 1 = 6.6 \times 10^{-25} \mathrm{GeV\,s}$$

This would give

$$\Gamma = 0.15 \times 10^{10} \mathrm{s}^{-1}$$

or a lifetime of 6×10^{-10}s, whereas the real value is 2.5×10^{-8}. This isn't terrible. We've missed some important factors, but the answer is indicative.

The same crude estimate can work for beta decay. Again, the decay rate needs to be very roughly

$$\Gamma = G_F^2 p^5$$

Neutron Decay

Let's try this for neutron decay. The energy release is $m_n - m_p - m_e = 0.78$ MeV. Let's take $p = 0.3$ MeV. Then

$$\Gamma = 10^{-10} \times 10^{-15} \times 3 \times 10^{-3}\ \mathrm{GeV} = 0.5 \times 10^{-3}\mathrm{s}^{-1}$$

that is, a lifetime of 2000 s, which is a fortuitously good estimate: $\tau_n = 886$ s.

Notice that the actual mass of the neutron doesn't enter, only the available energy. If the neutron were very much more massive, but the mass differences were unchanged, the lifetime would be nearly the same. That's why we didn't use the neutron mass in place of some power of p.

Scattering Cross Section (I)

- Want to measure cross section $\sigma \equiv \sigma_{A+B\to C+D}$
- $d\sigma/d\Omega$ defined as rate of scattering per beam particle and target particle per unit solid angle per sec:

$$rate = \Phi \frac{d\sigma}{d\Omega}$$

  where Φ is flux in # beam particles/area/sec
- Unit of σ: area

$$1 \text{ barn} = 10^{-28} \text{ m}^2 \quad \text{very big}$$

  mb, nb, pb, fb are typical units

Scattering Cross Section (II)

$$
\begin{aligned}
w_{fi} &= (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B)|T_{fi}|^2 \\
\text{Cross Section} &= \frac{w_{fi}}{\text{initial flux}} (\text{number of final states})
\end{aligned}
$$

- Number of scatters:

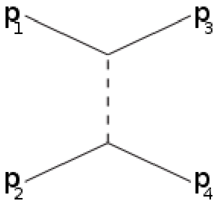
$$n_s = (n_b v_b) N_t \sigma$$

where v_b is relative velocity of beam and target, n_b is the beam particle density ($\#/\mathrm{m}^3$) and $N_t = n_t A_B \Delta L$ is the number of target particles within the beam area.

Cross Sections and Lorentz Invariants

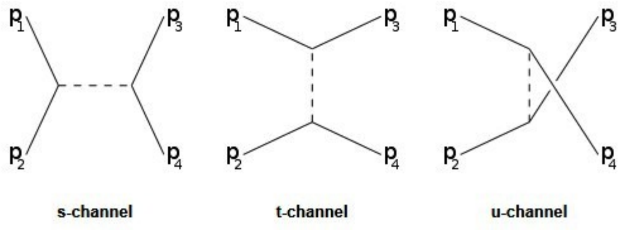
- Cross sections are easy to estimate at high energies, where we can ignore masses of scattered particles
- For $p_1 + p_2 \to p_3 + p_3$ the Mandelstam variables are

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &= (p_3 - p_1)^2 = (p_4 - p_2)^2 \\ u &= (p_3 - p_2)^2 = (p_4 + p_1)^2 \end{aligned}$$



- In all cases $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

s,t and u channel exchange



- We have seen that interactions occur via exchange of vector bosons
- This exchange can be an annihilation (s) can just be absorbtion and emission
- Describe in terms of the Mandelstam variables

$e^+e^- \to \mu^+\mu^-$

- Ignoring masses and factors like i, a propagator in the s channel give a factor $1/s$.
- Similarly, a propagator in the t channel gives a factor $1/t$. So for $e^+e^- \to \mu^-\mu^-$
$$T_{fi} \propto \alpha/s$$
- Now $\sigma \propto |\mathcal{M}|^2$, so since there are no other dimensionful variables
$$\sigma \approx \frac{\alpha^2}{s}$$
- The real answer is
$$\sigma = \frac{4\pi\alpha^2}{3s} = \frac{86.8 \text{ nb}}{s(\text{GeV}^2)}$$
- Since a barn is 10^{-24}cm^2, 1 nb$=10^{-33}$ cm^2. Luminosities of e^+e^- machines at $s = 100\text{GeV}^2$ are typically of order $10^{33}\text{s}^{-1}\text{cm}^{-2}$, giving about an event per second.

t-channel scattering

- What about $\mu^+e^- \to \mu^+e^-$ at some hypothetical storage ring.
- Here we can exchange a photon in the t channel. We need to consider the differential cross section

$$\frac{d\sigma}{d\Omega} \propto \frac{\alpha^2 s}{t^2}$$

- We have argued on dimensional grounds that the numerator is s, but it could also have been u^2/s. In fact, the numerator depends in detail on whether we are scattering spin-1/2 or spin-zero particles.
- Note that

$$t = -4p_{cm}^2 \sin^2\theta/2$$

where θ is the cm scattering angle. We see that that cross section becomes infinite in the forward direction. Indeed, naively the total cross section is infinite. We'll talk more about this in the context of Rutherford Scattering.

Scattering of Pointlike Particles

- Rutherford Scattering (spinless electron scattering from a static point charge) in lab frame:
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\frac{1}{2}\theta)}$$
where E is energy of incident electron, θ is scattering angle in the lab frame
- Mott Scattering: Taking into account statistics of identical spinless particles
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2(\frac{1}{2}\theta)}{4E^2 \sin^4(\frac{1}{2}\theta)}$$
- Elastic Scattering of spin-$\frac{1}{2}$ electron from pointlike spin-$\frac{1}{2}$ particle of mass M:
  - Scattering of electron from static charge changes angle but not energy
  - For target of finite mass M, final electron energy is
$$E' = \frac{E}{1 + \frac{2E}{M}\sin^2(\frac{1}{2}\theta)}$$
and the four-momentum transfer is
$$q^2 = -4EE' \sin^2(\frac{1}{2}\theta)$$
- The elastic scattering cross section is:
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \cos^2(\frac{1}{2}\theta)}{4E^2 \sin^4(\frac{1}{2}\theta)} \frac{E'}{E} \left[1 - \frac{q^2}{2M^2} \tan^2(\frac{1}{2}\theta)\right]$$

What Happens if the Target Particles Have Finite Size?

- Charge distribution $\rho(r)$: $\int \rho(r) d^3r = 1$
- Xcattering amplitude modified by a "Form Factor"

$$F(q^2) = \int d^3r e^{i\vec{q}\cdot\vec{r}} \rho(r)$$

So that the cross section is modified by a factor of $|F(q^2)|^2$

- Note: As $q^2 \to 0$, $F(q^2) \to 1$
- We therefore can Taylor expand

$$F(q^2) = \int d^3r \left(1 + i\vec{q}\cdot\vec{r} - \frac{1}{2}(\vec{q}\cdot\vec{r})^2 + ...\right) \rho(r)$$

Form Factors

- The first $\vec{q} \cdot \vec{r}$ term vanishes when we integrate

$$
\begin{aligned}
F(q^2) &= 1 - \frac{1}{2} \int r^2 dr d\cos\theta d\phi \rho(r) (qr)^2 \cos^2\theta \\
&= \frac{2\pi}{2} \int dr d\cos\theta q^2 r^4 \cos^2\theta \\
&= 1 - \frac{<r^2>}{4} q^2 \int \cos^2\theta d\cos\theta \\
&= 1 - \frac{<r^2>}{4} q^2 \left[\frac{\cos^3\theta}{3} \right]_{-1}^{1} \% \\
&= 1 - \frac{<r^2>}{6} q^2
\end{aligned}
$$

This F is called the "form factor"

- Thus, if we plot $\frac{d\sigma}{d\Omega} / \frac{d\sigma}{d\Omega}_{pointlike}$ vs $\tan^2 \frac{1}{2}\theta$ or vs q^2 we can measure the size of the proton

$<r^2>^{\frac{1}{2}} = 0.74 \pm 0.24 \times 10^{13}$ cm ~ 0.7 fm

(McAllister and Hofstadter, 1956)

See the next page for relevant plots

Hoffstader and McAllister's experimental setup and results

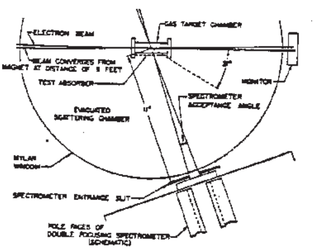
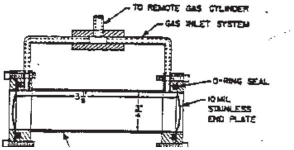


FIG. 2. Arrangement of parts in experiments on electron scattering from a gas target.



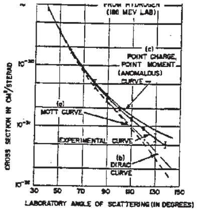


FIG. 5. Curve (a) shows the theoretical Mott curve for a spinless point proton. Curve (b) shows the theoretical curve for a point proton with the Dirac magnetic moment, curve (c) the theoretical curve for a point proton having the anomalous contribution in addition to the Dirac value of magnetic moment. The theoretical curves (b) and (c) are due to Rosenbluth.[8] The experimental curve falls between curves (b) and (c). This deviation from the theoretical curves represents the effect of a form factor for the proton and indicates structure within the proton, or alternatively, a breakdown of the Coulomb law. The best fit indicates a size of 0.70×10^{-13} cm.

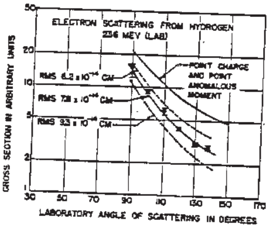


FIG. 6. This figure shows the experimental points at 236 Mev d the attempts to fit the shape of the experimental curve. The t fit lies near 0.78×10^{-13} cm.